Slender oscillating ships at zero forward speed

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The ship is assumed to be a slender body of revolution with its axis in the mean free surface and making periodic oscillations of small amplitude. The theory presented here is a generalization of the well-known slender-body theory of incompressible aerodynamics in which the fluid is externally unbounded. One version of that theory goes as follows: approximate the body by axial line-distributions of known point singularities (sources and multipoles), whose strength is to be determined; by means of the Fourier convolution theorem express the velocity potentials of these line distributions in terms of the Fourier transforms (in the axial direction) of the point-singularity potentials; expand these Fourier transforms in powers of the radius and retain only the leading terms (it is here that the slender-body assumption is introduced); by means of the Fourier convolution theorem interpret the resulting expressions. By this procedure it is not only shown that near the body the potential is two-dimensional harmonic in every plane normal to the axis; but also the interaction between sections is shown to be involved in the 'constant' term and to depend in an explicit manner on the coefficient functions, which can be found without difficulty by applying the prescribed boundary conditions. This foregoing procedure can be justified when the body is slender and sharply pointed.

In the present paper the same procedure is adapted to an oscillating surface ship at zero speed. The fluid is now bounded by the ship, and also by a horizontal plane (the mean free surface) on which a wave boundary condition must be applied. The point singularities are now wave sources and wave-free potentials, each satisfying the free-surface condition. The Fourier transforms of these singularities are found and are expanded near the axis; the expansions near the axis are the only parts of the argument that present any serious difficulty. When only the leading terms are retained and the results are interpreted by the convolution theorem, explicit two-dimensional potentials are again obtained.

It is assumed that the ratios (ship-radius/ship-length) and (ship-radius/wave-length) are small whereas the ratio (ship-length/wave-length) may have any value. Expressions are given which are valid according as this last ratio is not large or not small. The potential on the body is found, and forces, moments and wave damping are calculated. It is believed that the expansions can be extended with little trouble to certain other ranges of those ratios, to other cross-sections and to the boundary condition for ships moving at non-zero speeds.

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1. Introduction

In the present paper we shall be concerned with the fluid motion near an oscillating ship, and with the resulting hydrodynamical forces. In practice gravity effects (such as wave resistance) and viscous effects (such as skin friction) are both important and cannot be separated, but, since there is as yet no theory including both, we shall here neglect viscous effects; the motion is then irrotational and is described by a velocity potential. Inviscid-flow calculations are familiar in aerodynamics, and there their role in determining the total force is now quite well understood (see Thwaites 1960, Ch. 3). The motion of surface ships is in comparison much less tractable, because the motion of the free surface must be determined as part of the problem. This gives rise to a wave resistance even in an inviscid fluid. No mathematical progress can be made unless the equations can be linearized, for which purpose additional simplifying assumptions are made: only such bodies and ship motions are treated as will cause the resulting fluid motion to differ only slightly from a state of rest or of uniform flow.

For instance, we may consider a *thin-ship* model, in effect a vertical plate of small thickness moving horizontally along itself with uniform velocity (Michell 1898). Such a thin ship, of beam much smaller than draft and length, can be represented by a distribution of Kelvin wave sources over its mid-plane (for a review see Wehausen 1957). The oscillations of the Michell ship have recently been studied by Peters & Stoker (1957) who have confirmed that in the first approximation its virtual mass and wave damping both vanish. The Michell model is therefore inappropriate when virtual mass and wave damping are of interest, as in most problems of ship motion in waves.

To overcome this defect it is natural to consider *slender* ships, of comparable beam and draft, and of length much greater than either. Since evidently the disturbance tends to zero as beam and draft both tend to zero (when the ship contracts to a line), a scheme of linearization based on the thickness–length ratio is reasonable. A considerable amount of work has already been published on submerged slender ships, but it is generally realized that the motion near a submerged ship differs materially from the motion near a surface ship. Many of these calculations have been concerned with strip theories. It can be seen that near a slender body the motion in planes normal to the ship's axis is nearly plane irrotational and can, to a rough first approximation, be found without reference to the flow parallel to the axis. In the strip theories the three-dimensional motion is synthesized approximately by combining these two-dimensional motions, and various ways of doing this have been suggested. Havelock (1956) has compared two strip methods of calculating the wave damping of a submerged spheroid at zero speed. The first method consists in calculating the two-dimensional energy transfer per unit length from a circular cylinder of radius equal to the local radius, and integrating this along the length of the cylinder; this clearly cannot give a good approximation when the waves are long compared to the length of the ship. In the second method the spheroid is replaced by an axial distribution of wave dipoles, of strength depending on the local area of cross-section, and this gives good results provided only that the spheroid is sufficiently slender. Both these methods have been applied to surface ships at zero speed, the first by Tasai (1959), the second by Grim (1957, 1960) who points out that for surface ships of finite length the first method wrongly predicts an infinite virtual mass for infinite wave-length.

Apparently no attempt has hitherto been made to adapt for surface ships the techniques of the well-established slender-body theory for incompressible unbounded media (for an account see Thwaites 1960, Ch. 9, § 11). It is the purpose of the present paper to provide such an adaptation. One version of the theory for unbounded media proceeds in the following steps:

(1) Approximate the body by axial line-distributions of known point singularities (sources and multipoles), whose strength is to be determined.

(2) By means of the Fourier convolution theorem express the velocity potentials of these line distributions in terms of the Fourier transforms (in the axial direction) of the point-singularity potentials.

(3) Expand these Fourier transforms in powers of the radius (logarithms will also appear), and retain only the leading terms. (It is here that the slender-body assumption is used.)

(4) By means of the Fourier convolution theorem interpret the resulting expressions.

This procedure can be justified if the body is sharply pointed at the ends. (Since the problem for an unbounded incompressible medium is linear it should be possible to devise suitable end-corrections for blunt bodies, but this has apparently not yet been done.)

If co-ordinates are defined as in § 2 below and if the motion is symmetrical about $\theta = 0$ and $\theta = \frac{1}{2}\pi$, then in an unbounded medium the velocity potential is thus found near the body to have the approximate form

$$\phi(x, r\cos\theta, r\sin\theta) = 2a_0(x)\ln\frac{L}{r} + 2\sum_{1}^{\infty}\frac{a_n(x)}{2n}\frac{\cos 2n\theta}{r^{2n}} \tag{1.1}$$

$$-\int_{-\infty}^{x}\frac{da_0(\xi)}{d\xi}\ln\left(\frac{L}{2|x-\xi|}\right)d\xi + \int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}\ln\left(\frac{L}{2|x-\xi|}\right)d\xi, \tag{1.2}$$

where the terms (1.1) are clearly plane harmonic in every plane $x = \text{const.}$ (and where the coefficient functions can be found from the boundary condition), while the terms (1.2) describe the interaction between sections, which is the principal object of the investigation. The same result has also been obtained in other ways, by Ward (1955) without the use of Fourier transforms, and by Tuck (1962) using a procedure of matching inner and outer expansions which is particularly convenient for non-circular sections.

The foregoing procedure will here be adapted to a periodically oscillating slender surface ship at zero forward speed. The velocity potential is now defined in that part of the lower half-space $y > 0$ lying outside the ship. On the mean free surface $y = 0$ a boundary condition must be applied. Accordingly for step (1) the point singularities will be chosen to satisfy this free-surface condition and we shall in fact take them to be wave sources and wave-free potentials. For step (2) we shall need their Fourier transforms, which are obtained in the Appendix at the end of the paper. For step (3) we shall need the small-radius expansions of these

Fourier transforms, also given in the Appendix. The transforms of the wave-free potentials are easily expanded, but the transform of the wave source is much more complicated and its expansion is the most difficult part of the problem. For step (4) we shall obtain the transforms of the leading terms in the expansion.

Expansions in terms of a wave source and an infinite set of wave-free potentials were introduced for plane problems by Ursell (1949*a*, *b*) and for the oscillating sphere by Havelock (1955). The three-dimensional expansion used here generalizes Havelock's work and was proposed by Grim (1957, 1960), who did not succeed in going beyond step (1) and thus obtained neither an expansion analogous to (1.1), (1.2) nor the interaction between sections. He observed, however, that when this interaction is negligible ($KL \gg 1$) the local source strength must be the same as for an infinitely long cylinder, of appropriate cross-section, and he suggested an iterative scheme of approximation.

A very interesting alternative treatment both for zero and non-zero speeds has recently been proposed by Vossers (1962). By means of Green's theorem he obtains a linear Fredholm integral equation of the second kind for the potential on the ship where the normal velocity is prescribed. If now the variation in the axial direction is treated as slow, then the integrals can be approximated by simpler integrals; in particular the kernel of the equation may be expected to reduce to the kernel appropriate to a plane problem, to which known techniques are applicable.

Vossers's results require difficult approximations for quadruple integrals, and his resulting integral equation (Ch. 5) is simpler than other integral equations previously obtained for plane problems (e.g. Ursell 1953), which suggests the possibility of errors. Nevertheless, it is felt that the method can be made to work, and that it has many advantages, even if the results so far obtained must be regarded as provisional.

We note that at zero forward speed our problem is linear provided only that the amplitude of oscillation is sufficiently small. Throughout the present paper the body will be assumed to be sharply pointed, and difficulties near the ends will be ignored. Only bodies of revolution with their axis in the surface will be considered. It will be assumed that the ratios (ship-radius/ship-length) and (ship-radius/wave-length) are small whereas the ratio (ship-length/wave-length) may have any value. Expressions will be given which are valid according as this last ratio is not large or not small. The potential on the body will be found; and forces, moments and wave damping will be calculated.

2. Expansion of the velocity potential

It will be supposed that the mean position of the axis of the slender body of revolution lies in the mean free surface $y = 0$, along the segment $-\frac{1}{2}L \leqslant x \leqslant \frac{1}{2}L$ of the x-axis, and that y increases with depth. The z-co-ordinate is measured horizontally at right angles to the x-axis. Cylindrical polar co-ordinates are defined by $y = r\cos\theta$, $z = r\sin\theta$. The equation of the mean position of the body is taken to be $r = r_0(x)$, where $r_0'(x) = dr_0(x)/dx$ is small, and where it is supposed initially that r_0 and all its derivatives vanish at the ends $x = \pm\frac{1}{2}L$. (This restriction will be discussed below, near the end of § 2.) We consider simple harmonic forced motions of heaving and pitching of period $2\pi/\sigma$, at zero mean speed. Since the amplitude is small, the boundary condition on the body may be applied at the mean position of the body. Then the velocity potential $\phi(x,y,z)\,e^{-i\sigma t}$ satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,y,z) = 0 \tag{2.1}$$

in the fluid. The boundary condition on the body $r = r_0(x)$ is

$$\frac{\partial \phi}{\partial n} = \frac{v_0+\omega_0 x+\omega_0 r_0(x)\,r_0'(x)}{\{1+(r_0')^2\}^{\frac{1}{2}}}\cos\theta, \tag{2.2}$$

where $(\partial\phi/\partial n)\,e^{-i\sigma t}$ is the velocity component normal to the body, $v_0\,e^{-i\sigma t}$ is the forced velocity of heaving parallel to the y-axis, and $\omega_0\,e^{-i\sigma t}$ is the forced angular velocity of pitching about the z-axis. The boundary condition (2.2) may be written in the form

$$\begin{aligned}\partial\phi/\partial r - r_0'(x)\,\partial\phi/\partial x &= (v_0+\omega_0 x+\omega_0 r_0 r_0')\cos\theta \\ &= V(x)\cos\theta, \quad \text{say,}\end{aligned} \tag{2.3}$$

where the second term on the left-hand side is small compared with the first. We shall accordingly replace (2.3) temporarily by the boundary condition

$$\partial\phi/\partial r = V(x)\cos\theta \quad \text{on} \quad r = r_0(x), \tag{2.4}$$

where $V(x)$ is a prescribed function. The boundary condition on the mean free surface $y = 0$ is

$$K\phi + \partial\phi/\partial y = 0, \tag{2.5}$$

where $2\pi/K = 2\pi g/\sigma^2$ is the wave-length of waves of period $2\pi/\sigma$. This linearized condition and condition (2.2) are valid when the amplitudes of oscillation are small compared with all other lengths involved in the problem; the body need not be slender. The boundary condition at infinity is a radiation condition: the waves travel towards infinity,

$$R^{\frac{1}{2}}(\partial\phi/\partial R - iK\phi) \to 0 \tag{2.6}$$

as $R = (x^2+y^2+z^2)^{\frac{1}{2}} \to \infty$.

We shall construct a velocity potential by the superposition of potential functions each satisfying identically all the boundary conditions except (2.4) which will then be satisfied to a sufficient approximation by a suitable choice of coefficient functions. We shall see that (except possibly near the ends) the flow can be approximately generated by wave singularities distributed along the axis of the body. Similar constructions are familiar in the flow past a slender body in an unbounded medium and are known not to be exact except possibly for a restricted class of smooth bodies. Our expansion is

$$\phi(x,y,z) = \int_{-\infty}^{\infty} a_0(\xi)\,\phi_0(x-\xi,y,z)\,d\xi \tag{2.7}$$

$$+\sum_{1}^{\infty}\int_{-\infty}^{\infty} a_n(\xi)\,\phi_n(x-\xi,y,z)\,d\xi,\dagger \tag{2.8}$$

† There is an interesting difficulty connected with the completeness of this expansion at a large distance from the body. Consider the analytic continuations of $\phi(x, y, z)$ into the half-space $y < 0$. Since the wave-free potentials are single-valued, the terms (2.8) are also single-valued, and so is their sum when $y^2+z^2 > \{\max r_0(x)\}^2$. And the term (2.7) is single-valued in $y < 0$ except for a cut along the plane $z = 0$. But it can be shown that the potential $\phi(x, y, z)$ cannot in general be continued to points vertically above the body. Thus the expansion (2.7)–(2.8) cannot be complete. It is interesting to remark that the transform $\Phi_0(k, y, z)$ can for every k be continued into $y^2+z^2 > \{\max r_0(x)\}^2$ except for a cut along $z = 0$. This is somewhat different from the corresponding difficulty in ordinary slender-body theory, which relates to analytic continuation into the body.

where the coefficient-functions $a_0(\xi)$, $a_n(\xi)$ vanish outside $|\xi| \leqslant \frac{1}{2}L$. The potential ϕ_0 in (2.7) is a wave source at the origin (Thorne 1953, p. 712)

$$\phi_0(x,y,z) = \oint_0^\infty \frac{k'}{k'-K} e^{-k'y} J_0\{k'(x^2+z^2)^{\frac{1}{2}}\} dk'; \tag{2.9}$$

the path of integration passes below the pole $k' = K$ in order that (2.9) may satisfy the radiation condition (2.6). The potentials ϕ_n in (2.8) are wave-free potentials singular at the origin

$$\phi_n(x,y,z) = \frac{P_{2n}(y/R)}{R^{2n+1}} + \frac{K}{2n}\frac{P_{2n-1}(y/R)}{R^{2n}} \quad (n = 1, 2, 3, \ldots), \tag{2.10}$$

where P_m is the Legendre polynomial of degree m, and $R = (x^2+y^2+z^2)^{\frac{1}{2}}$. It is known from the work of Havelock (1955) on the heaving half-immersed sphere that (2.10) satisfies Laplace's equation (2.1) and the free-surface condition (2.5); the radiation condition (2.6) is satisfied trivially. Each of the singularities (2.9), (2.10) has axial symmetry about the (vertical) y-axis. We could write down-wave sources with other symmetries and consider line distributions analogous to (2.7) and (2.8), but we shall see that (2.7) and (2.8) are sufficient approximations for our purpose. The expansion (2.7)–(2.8) was first proposed by Grim (1957, 1960), but our determination of the coefficients and our conclusions will be different.

We shall assume initially that not only $a_0(x)$ and $a_n(x)$ but also all their differential coefficients are continuous when $|x| \leqslant \frac{1}{2}L$, and that they vanish when $x = \pm\frac{1}{2}L$ and when $|x| > \frac{1}{2}L$. (This corresponds to a body with very sharply pointed ends, see the discussion near the end of § 2.) Then the Fourier transforms

$$A_n(k) = \int_{-\infty}^{\infty} a_n(\xi)\, e^{ik\xi}\, d\xi$$

tend to zero rapidly as $|k|$ tends to infinity (Lighthill 1958, theorem 2), and by the convolution theorem for Fourier transforms we have from (2.7) and (2.8) the equation

$$\phi_n(x,y,z) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} A_n(k)\,\Phi_n(k,y,z)\,e^{-ikx}\,dk, \tag{2.11}$$

where $\Phi_n(k,y,z) = \int_{-\infty}^{\infty} \phi_n(x,y,z)\,e^{ikx}\,dx$ are the Fourier transforms in the x-direction, which clearly must satisfy

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right)\Phi_n(k,y,z) = 0. \tag{2.12}$$

As was explained in the introduction, we shall now expand Φ_0 and Φ_n near ($y = 0$, $z = 0$), retain only the leading terms and then apply the inverse Fourier transformation. The expansion of Φ_0 is the most difficult part of the argument, while the expansion of Φ_n is straightforward. The Fourier inversion will lead to new functions s_1, s_2 and s_3 which will appear in the interaction term. It is shown in the Appendix at the end of this paper that

$$\begin{aligned}\Phi_0(k,y,z;K) &= 2\int_0^\infty \frac{|k|\cosh\mu}{|k|\cosh\mu - K}e^{-|k|y\cosh\mu}\cos(|k|z\sinh\mu)\,d\mu \\ &= \begin{Bmatrix}(\pi i-\alpha)\coth\alpha \\ (\pi-\alpha^*)\cot\alpha^*\end{Bmatrix}\left(2I_0(|k|r)+4\sum_1^\infty(-1)^m I_m(|k|r)\cos m\theta\begin{Bmatrix}\cosh m\alpha \\ \cos m\alpha^*\end{Bmatrix}\right) \\ &\quad + 2K_0(|k|r)+4\sum_1^\infty(-1)^{m-1}\left[\frac{\partial}{\partial\nu}(I_\nu(|k|r)\cos\nu\theta)\right]_{\nu=m}\begin{Bmatrix}\sinh m\alpha\coth\alpha \\ \sin m\alpha^*\cot\alpha^*\end{Bmatrix},\end{aligned} \tag{2.13}$$

where the upper or the lower expression is applicable according as

$$\frac{K}{|k|} = \begin{Bmatrix}\cosh\alpha > 1 \\ \cos\alpha^* < 1\end{Bmatrix}. \tag{2.14}$$

Note that $\partial(I_\nu(|k|r)\cos\nu\theta)/\partial\nu = \cos\nu\theta(\partial I_\nu/\partial\nu) - \theta\sin\nu\theta\, I_\nu$ is not a single-valued function of θ; a cut along $\theta = \pm\pi$ is needed. In these equations, I_ν and K_0 are Bessel functions of imaginary argument (Erdélyi 1953, p. 9),

$$I_\nu(\zeta) = \sum_{s=0}^\infty \frac{(\frac{1}{2}\zeta)^{\nu+2s}}{s!\Gamma(\nu+s+1)}, \tag{2.15}$$

$$\frac{\partial I_\nu(\zeta)}{\partial\nu} = \sum_{s=0}^\infty \frac{(\frac{1}{2}\zeta)^{\nu+2s}}{s!\Gamma(\nu+s+1)}\{\ln\tfrac{1}{2}\zeta - \psi(\nu+s+1)\}, \tag{2.16}$$

$$K_0(\zeta) = -\left(\frac{\partial I_\nu(\zeta)}{\partial\nu}\right)_{\nu=0}, \quad \psi(\zeta) = \frac{d}{d\zeta}\ln\Gamma(\zeta), \tag{2.17}$$

$$\psi(1) = -\gamma, \quad \psi(m+1) = -\gamma+\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m}, \tag{2.18}$$

when m is an integer, and $\gamma = 0{\cdot}5772\ldots$ is Euler's constant.

We can use these power series to expand $\Phi_0(k,y,z)$ for small r:

$$\begin{aligned}\Phi_0(|k|, r\cos\theta, r\sin\theta; K) = {} & -2\ln Kr + 2Kr\ln Kr\cos\theta - 2Kr\,\theta\sin\theta \\ & - 2Kr\cos\theta - 2\gamma(1-Kr\cos\theta) \\ & + 2(1-Kr\cos\theta)\{F_1(|k|/K)+F_2(|k|/K)\} \\ & + O(r^2\ln r),\end{aligned} \tag{2.19}$$

where

$$F_1\left(\frac{|k|}{K}\right) = \begin{Bmatrix}i\pi\coth\alpha \\ \pi\cot\alpha^*\end{Bmatrix}, \tag{2.20}$$

and

$$F_2\left(\frac{|k|}{K}\right) = \begin{Bmatrix}\ln(2\cosh\alpha)-\alpha\coth\alpha \\ \ln(2\cos\alpha^*)-\alpha^*\cot\alpha^*\end{Bmatrix}. \tag{2.21}$$

Thus

$$\begin{aligned}&\int_{-\infty}^\infty a_0(\xi)\,\phi_0(x-\xi,y,z)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty A_0(k)\,\Phi_0(k,y,z)\,e^{-ikx}\,dk \\ &\doteqdot \frac{1}{2\pi}\int_{-\infty}^\infty A_0(k)\{-2\ln Kr + 2Kr\ln Kr\cos\theta - 2Kr\,\theta\sin\theta \\ &\qquad\qquad - 2Kr\cos\theta - 2\gamma(1-Kr\cos\theta)\}\,e^{-ikx}\,dk \qquad (2.22) \\ &\quad + 2(1-Kr\cos\theta)\frac{1}{2\pi}\int_{-\infty}^\infty A_0(k)\left\{F_1\left(\frac{|k|}{K}\right)+F_2\left(\frac{|k|}{K}\right)\right\}e^{-ikx}\,dk \qquad (2.23) \\ &\quad + O(r^2\ln r),\end{aligned}$$

where we are using the assumption that $A_0(k)$ tends rapidly to zero as $|k|$ tends to infinity to ensure that all integrals in (2.22) are convergent. In (2.22) we now use the Fourier inversion formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_0(k)\,e^{-ikx}\,dk = a_0(x), \tag{2.24}$$

and in (2.23) we use the convolution theorem to express the integral in terms of $a_0(x)$. Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_0(k)\,F_1\left(\frac{|k|}{K}\right)e^{-ikx}\,dk = \int_{-\infty}^{\infty} a_0(\xi)\,Kf_1\{K(x-\xi)\}\,d\xi,$$

where

$$\begin{aligned} Kf_1(K\zeta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1\left(\frac{|k|}{K}\right)e^{-ik\zeta}\,dk \\ &= \frac{1}{\pi}\int_0^K \frac{i\pi K}{(K^2-k^2)^{\frac{1}{2}}}\cos k\zeta\,dk + \frac{1}{\pi}\int_K^{\infty}\frac{\pi K}{(k^2-K^2)^{\frac{1}{2}}}\cos k\zeta\,dk \\ &= \tfrac{1}{2}i\pi K H_0^{(1)}(K|\zeta|) \end{aligned}$$

from Watson (1944, p. 48, equation (2), and p. 170, equation (4)); here $H_0^{(1)}$ is the Hankel function J_0+iY_0. Thus we see that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_0(k)\,F_1\left(\frac{|k|}{K}\right)e^{-ikx}\,dk = \tfrac{1}{2}i\pi K\int_{-\infty}^{\infty} a_0(\xi)\,H_0^{(1)}(K|x-\xi|)\,d\xi. \tag{2.25}$$

We also write down an alternative form for (2.25). We have

$$F_1(|\eta|) \sim i\pi + \tfrac{1}{2}i\pi|\eta|^2 \quad \text{as} \quad \eta \to 0,$$

whence

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} A_0(k)\,F_1\left(\frac{|k|}{K}\right)e^{-ikx}\,dk \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_0(k)\,i\pi\,e^{-ikx}\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty}(-ik)\,A_0(k)\,\frac{F_1(|k|/K)-i\pi}{(-ik)}\,e^{-ikx}\,dk. \end{aligned} \tag{2.26}$$

By use of the obvious relation

$$-ikA_0(k) = \int_{-\infty}^{\infty}\frac{da_0(x)}{dx}\,e^{ikx}\,dx, \tag{2.27}$$

we see that (2.26) is equal to

$$i\pi a_0(x) + \int_{-\infty}^{\infty}\frac{da_0(\xi)}{d\xi}\,g_1\{K(x-\xi)\}\,d\xi, \tag{2.28}$$

where by definition

$$\begin{aligned} g_1(K\zeta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_1(|k|/K)-i\pi}{(-ik)}\,e^{-ik\zeta}\,dk \\ &= \frac{1}{\pi}\int_0^{\infty}\frac{F_1(\eta)-i\pi}{\eta}\sin(\eta K\zeta)\,d\eta, \end{aligned} \tag{2.29}$$

which is clearly an odd function of $K\zeta$. We may therefore write

$$g_1(K\zeta) = -\tfrac{1}{2}s_1(|K\zeta|)\,\mathrm{sgn}\,K\zeta, \tag{2.30}$$

where $\operatorname{sgn}\eta = \pm 1$ according as $\eta \gtrless 0$, and we obtain for (2.26) the alternative form

$$i\pi a_0(x) - \frac{1}{2}\int_{-\infty}^{x} \frac{da_0(\xi)}{d\xi} s_1(K\,|x-\xi|)\,d\xi + \frac{1}{2}\int_{x}^{\infty} \frac{da_0(\xi)}{d\xi} s_1(K\,|x-\xi|)\,d\xi. \quad (2.31)$$

It can be shown from (2.29) and (2.48) that $s_1(K\zeta) \to 0$ as $K\zeta \to +\infty$, and it can thence be shown that the form (2.31) is appropriate when KL is large (see § 4 below) but we do not investigate the properties of the function s_1 in detail here.

The other integral $(1/2\pi)\int A_0(k)\,F_2(|k|/K)\,e^{-ikx}\,dk$ occurring in (2.23) can be treated similarly. The function

$$F_2(\zeta) \equiv -\ln\tfrac{1}{2}\zeta + (1-\zeta^2)^{-\frac{1}{2}}\ln\left[\zeta/\{1+(1-\zeta^2)^{\frac{1}{2}}\}\right] \quad (2.32)$$

is regular in the positive quadrant except at $\zeta = 0$; it is easy to see that $\zeta = 1$ is not a singularity. As $\zeta \to 0$, $F_2(\zeta) \sim \frac{1}{2}\zeta^2 \ln\zeta$; as $\zeta \to \infty$, $F_2(\zeta) \sim -\ln\frac{1}{2}\zeta$. Thus the integral is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (-ik)\,A_0(k)\,\frac{F_2(|k|/K)}{(-ik)}\,e^{-ikx}\,dk$$

$$= \int_{-\infty}^{\infty} \frac{da_0(\xi)}{d\xi}\,g_2\{K(x-\xi)\}\,d\xi, \quad (2.33)$$

where by definition

$$g_2(K\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F_2(|k|/K)}{(-ik)}\,e^{-ik\zeta}\,dk$$

$$= \frac{1}{\pi}\int_{0}^{\infty} F_2(\eta)\,\frac{\sin(\eta K\zeta)}{\eta}\,d\eta, \quad (2.34)$$

which is clearly an odd function of $K\zeta$. In deriving an alternative form for (2.34) it is therefore sufficient to consider positive values of $K\zeta$. The integral (2.34) is the imaginary part of

$$\frac{1}{\pi}\int_{0}^{\infty} k^{-1}F_2(k/K)\,e^{ik\zeta}\,dk$$

which can be transformed by rotating the line of integration to the positive imaginary k-axis. We find that

$$\frac{1}{\pi}\int_{0}^{\infty} k^{-1}F_2(k/K)\,e^{ik\zeta}\,dk$$

$$= \frac{1}{\pi}\int_{0}^{\infty} w^{-1}F_2(w)\,e^{iw\,K\zeta}\,dw$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \left\{-\ln\tfrac{1}{2}w - \tfrac{1}{2}\pi i + (1+w^2)^{-\frac{1}{2}}\left(\ln\frac{w}{1+(1+w^2)^{\frac{1}{2}}} + \tfrac{1}{2}\pi i\right)\right\} e^{-w\,K\zeta}\,\frac{dw}{w}$$

of which the imaginary part is

$$g_2(K\zeta) = -\frac{1}{2}\int_{0}^{\infty} \{1-(1+w^2)^{-\frac{1}{2}}\}\,e^{-w\,K\zeta}\,\frac{dw}{w}$$

when $K\zeta > 0$. Thus finally from (2.34)

$$g_2(K\zeta) = -\tfrac{1}{2}s_2(|K\zeta|)\operatorname{sgn}K\zeta, \quad (2.35)$$

where by definition

$$s_2(|\eta|) = \int_{0}^{\infty} \{1-(1+w^2)^{-\frac{1}{2}}\}\,e^{-w|\eta|}\,\frac{dw}{w} \quad (2.36)$$

and $\operatorname{sgn}\eta = \pm 1$ according as $\eta \gtrless 0$. It can be shown that

$$s_2(|\eta|) \doteqdot \tfrac{1}{2}|\eta|^{-2} \quad \text{as} \quad |\eta| \to \infty$$

and
$$s_2(|\eta|) = -\ln|\eta| - \gamma - \ln 2 + s_3(|\eta|), \quad \text{say}, \tag{2.37}$$

where by definition

$$s_3(|\eta|) = \int_0^\infty \frac{1-e^{-w|\eta|}}{w(1+w^2)^{\frac{1}{2}}}\,dw, \tag{2.38}$$

which tends to 0 as η tends to 0; see also (2.49). It follows from (2.33) and (2.35) that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_0(k)\,F_2\left(\frac{|k|}{K}\right)e^{-ikx}\,dk$$
$$= -\frac{1}{2}\int_{-\infty}^{x}\frac{da_0(\xi)}{d\xi}s_2(K|x-\xi|)\,d\xi + \frac{1}{2}\int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}s_2(K|x-\xi|)\,d\xi, \tag{2.39}$$

where s_2 is defined by (2.36). This completes the discussion of the axial wave-source distribution.

Turning now to the line distributions of wave-free potentials, we note (see Appendix at the end) that

$$\Phi_n(k,y,z;K) = \frac{2}{(2n)!}|k|^{2n-1}\int_0^\infty (K+|k|\cosh\mu)(\cosh\mu)^{2n-1}$$
$$\times\, e^{-|k|y\cosh\mu}\cos(kz\sinh\mu)\,d\mu. \tag{2.40}$$

Since
$$(\cosh\mu)^{2n-1} = (\tfrac{1}{2})^{2n-2}\sum_{s=0}^{s=n-1}\frac{(2n-1)!}{s!\,(2n-1-s)!}\cosh(2n-2s-1)\mu,$$

and
$$K_m(|k|\,r)\cos m\theta = \int_0^\infty \cosh m\mu\, e^{-|k|y\cosh\mu}\cos(kz\sinh\mu)\,d\mu,$$

where $K_m(\zeta)$ is a Bessel function of imaginary argument (Erdélyi 1953, p. 9), we may write

$$\Phi_n(k,y,z) = \frac{1}{(2n)!}|k|^{2n-1}(\tfrac{1}{2})^{2n-2}\sum_0^{n-1}\frac{(2n-1)!}{s!\,(2n-1-s)!}\{|k|\,K_{2n-2s}(|k|\,r)\cos(2n-2s)\theta$$
$$+|k|\,K_{2n-2s-2}(|k|\,r)\cos(2n-2s-2)\theta + 2K\,K_{2n-2s-1}(|k|\,r)\cos(2n-2s-1)\theta\}; \tag{2.41}$$

the leading terms in the expansion near $r = 0$ are

$$\frac{1}{(2n)!}(\tfrac{1}{2})^{2n-2}\left\{|k|\,\frac{1}{2}\frac{(2n-1)!}{(\frac{1}{2}|k|\,r)^{2n}}\cos 2n\theta + \frac{K(2n-2)!}{(\frac{1}{2}|k|\,r)^{2n-1}}\cos(2n-1)\theta + O\left(\frac{\ln r}{r^{2n-2}}\right)\right\}$$

and so

$$\Phi_n(k,y,z) \doteqdot \frac{2}{2n}\left(\frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\,\frac{\cos(2n-1)\theta}{r^{2n-1}}\right) + O\left(\frac{\ln r}{r^{2n-2}}\right), \tag{2.42}$$

with a smaller error when $n > 1$. It is seen that near $r = 0$ the functions Φ_n are nearly independent of k, and nearly equal to the wave-free potentials for the plane circle problem (Ursell 1949*a*, p. 223). It follows that

$$\int_{-\infty}^{\infty} a_n(\xi)\,\phi_n(x-\xi,y,z)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_n(k)\,\Phi_n(k)\,e^{-ikx}\,dk$$
$$\doteqdot \frac{2}{2n}a_n(x)\left(\frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\,\frac{\cos(2n-1)\theta}{r^{2n-1}}\right). \tag{2.43}$$

We can now write down the total approximate potential as the sum of (2.22), (2.23) and (2.43) where (2.23) is given by (2.25) and (2.39):

$$\phi(x,y,z) = a_0(x)\{-2\ln Kr + 2Kr\ln Kr\cos\theta - 2Kr\,\theta\sin\theta - 2Kr\cos\theta - 2\gamma(1-Kr\cos\theta) + O(r^2\ln r)\} \tag{2.44}$$

$$+2(1-Kr\cos\theta)\,\tfrac{1}{2}i\pi K\int_{-\infty}^{\infty} a_0(\xi)\,H_0^{(1)}(K|x-\xi|)\,d\xi + O(r^2) \tag{2.45}$$

$$+2(1-Kr\cos\theta)\left\{-\frac{1}{2}\int_{-\infty}^{x}\frac{da_0(\xi)}{d\xi}s_2(K|x-\xi|)\,d\xi + \frac{1}{2}\int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}s_2(K|x-\xi|)\,d\xi + O(r^2)\right\} \tag{2.46}$$

$$+2\sum_1^{\infty}\frac{a_n(x)}{2n}\left\{\frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\frac{\cos(2n-1)\theta}{r^{2n-1}} + O\left(\frac{\ln r}{r^{2n-2}}\right)\right\}, \tag{2.47}$$

where $s_2(|\eta|)$ is given by (2.36); see also §4 below.

It can be shown that, for $\zeta > 0$,

$$s_1(\zeta) = 2i\int_1^{\infty} e^{iw\zeta}\frac{dw}{w(w^2-1)^{\frac{1}{2}}} = \pi i - \pi i\int_0^{\zeta} J_0(\zeta')\,d\zeta' + \pi\int_0^{\zeta} Y_0(\zeta')\,d\zeta', \tag{2.48}$$

and

$$s_2(\zeta) = -\int_0^{\infty} e^{-w\zeta}\left(1-\frac{1}{(w^2+1)^{\frac{1}{2}}}\right)dw = -\ln 2\zeta - \gamma + \tfrac{1}{2}\pi\int_0^{\zeta}\mathbf{H}_0(\zeta')\,d\zeta' - \tfrac{1}{2}\pi\int_0^{\zeta} Y_0(\zeta')\,d\zeta'. \tag{2.49}$$

Here the notation of Watson (1944) and Erdélyi (1953) has been used: J_0 and Y_0 are Bessel functions order 0, and

$$\mathbf{H}_0(\zeta) = \sum_0^{\infty}(-1)^m\frac{(\frac{1}{2}\zeta)^{2m+1}}{\{\Gamma(m+\frac{3}{2})\}^2}$$

is the Struve function of order 0.

It has so far been assumed that $a_0(x)$ and $a_n(x)$ are infinitely differentiable, but in fact the full force of this assumption was not needed to derive the formula (2.44)–(2.47). Sufficient conditions for these will not be discussed in detail here, but it can be shown that it would be sufficient to assume, e.g. that $|kL|^3 A_0(k)$ and $|kL|^3 A_n(k)$ remain bounded as $k \to \pm\infty$, in which case this formula is obtained with slightly larger error terms. And these conditions on the Fourier transforms are satisfied if, e.g. $a_0(x)$, $a_n(x)$, $a_0'(x)$, $a_n'(x)$ are all continuous in the interval $-\frac{1}{2}L \leqslant x \leqslant \frac{1}{2}L$ including the end-points (Lighthill 1958, p. 56, equations (30) and (34)). There are additional logarithmic terms if a_0' and a_n' are discontinuous at the end-points, as is well known in slender-body theory for an unbounded medium (Thwaites 1960, p. 388, equation (69)). Such terms indicate a failure of the slender-body approximation, requiring further investigation. Stricter conditions are needed if $\partial\phi/\partial x$ is also to be bounded at the ends, but overall quantities like total force and moment are not likely to be seriously in error even if these conditions are not satisfied. We now return to the approximation (2.44)–(2.47) for the potential.

The next step must be to determine the coefficient functions $a_0(x)$, $a_n(x)$. It will appear that these take different forms according as KL is large or small, both forms being applicable when KL is moderate. For large KL it seems physically reasonable to expect that there should be a strip theory derivable from two-dimensional solutions, and this will be confirmed in § 4. The case of small KL is more interesting, and the solution reduces in the limit $KL \to 0$ to slender-body theory for an unbounded medium. This case will be studied in some detail in § 3. Forces, moments and wave-damping will be briefly discussed in § 5.

3. The boundary-value problem for small and moderate values of KL

Let us examine the behaviour of the expression (2.44)–(2.47) for the potential as $K \to 0$ formally. Then clearly $|(2.44)| \to \infty$; and in (2.46) the function $s_2 \to \infty$, see (2.37), and so $|(2.46)| \to \infty$. The sum of (2.44) and (2.46) remains finite, however. We rewrite the potential in terms of the function s_3 defined by (2.38),

$$\phi(x,y,z) = a_0(x)\{-2\ln(r/L_1) + 2Kr\ln(r/L)\cos\theta - 2Kr\,\theta\sin\theta - 2Kr\cos\theta + O(r^2\ln r)\} \quad (3.1)$$

$$+2(1-Kr\cos\theta)\,\tfrac{1}{2}i\pi K\int_{-\infty}^{\infty} a_0(\xi)\,H_0^{(1)}(K|x-\xi|)\,d\xi + O(r^2) \quad (3.2)$$

$$+2(1-Kr\cos\theta)\left\{-\frac{1}{2}\int_{-\infty}^{x}\frac{da_0(\xi)}{d\xi}\ln\left(\frac{L_1}{2|x-\xi|}\right)d\xi + \frac{1}{2}\int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}\ln\left(\frac{L_1}{2|x-\xi|}\right)d\xi\right\} \quad (3.3)$$

$$+2(1-Kr\cos\theta)\left\{-\frac{1}{2}\int_{-\infty}^{x}\frac{da_0(\xi)}{d\xi}s_3(K|x-\xi|)\,d\xi + \frac{1}{2}\int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}s_3(K|x-\xi|)\,d\xi + O(r^2)\right\} \quad (3.4)$$

$$+2\sum_{1}^{\infty}\frac{a_n(x)}{2n}\left\{\frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\frac{\cos(2n-1)\theta}{r^{2n-1}} + O\left(\frac{\ln r}{r^{2n-2}}\right)\right\}. \quad (3.5)$$

In this expression the length L_1 is arbitrary, and when KL is small or moderate it is convenient to take $L_1 = L$, as we shall henceforth do. As a check, if we let KL tend to 0 we find that

$$\phi \to -2a_0(x)\ln(r/L) + 2\left\{-\frac{1}{2}\int_{-\infty}^{x}\frac{da_0(\xi)}{d\xi}\ln\left(\frac{L}{2|x-\xi|}\right)d\xi + \frac{1}{2}\int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}\ln\left(\frac{L}{2|x-\xi|}\right)d\xi\right\} + 2\sum_{1}^{\infty}\frac{a_n(x)}{2n}\frac{\cos 2n\theta}{r^{2n}}, \quad (3.6)$$

a finite limit corresponding to the boundary condition $\partial\phi/\partial y = 0$ on $y = 0$, which agrees with the expression quoted in § 1 above. Returning now to the expression (3.1)–(3.5), we shall find the coefficient functions $a_0(x)$, $a_n(x)$ in terms of the prescribed radial-velocity distribution (2.4),

$$\partial\phi/\partial r = V(x)\cos\theta \quad \text{on} \quad r = r_0(x).$$

Let us now introduce the linear operator

$$\mathscr{L}(a_0(x)) = \tfrac{1}{2}i\pi K \int_{-\infty}^{\infty} a_0(\xi)\, H_0^{(1)}(K\,|x-\xi|)\,d\xi$$
$$+\left\{-\frac{1}{2}\int_{-\infty}^{x} \frac{da_0(\xi)}{d\xi}\ln\left(\frac{L}{2\,|x-\xi|}\right)d\xi + \frac{1}{2}\int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}\ln\left(\frac{L}{2\,|x-\xi|}\right)d\xi\right\}$$
$$+\left\{-\frac{1}{2}\int_{-\infty}^{x}\frac{da_0(\xi)}{d\xi}s_3(K\,|x-\xi|)\,d\xi + \frac{1}{2}\int_{x}^{\infty}\frac{da_0(\xi)}{d\xi}s_3(K\,|x-\xi|)\,d\xi\right\}. \quad (3.7)$$

When KL is small or moderate it is easy to see that

$$|\mathscr{L}(a_0(x))| = O\,|a_0(x)|.$$

Then (3.1)–(3.5) can be written more briefly as

$$\phi = a_0(x)\{-2\ln(r/L) + 2Kr\ln(r/L)\cos\theta - 2Kr\,\theta\sin\theta - 2Kr\cos\theta\}$$
$$+2(1-Kr\cos\theta)\mathscr{L}(a_0)$$
$$+2\sum_{1}^{\infty}\frac{a_n(x)}{2n}\left(\frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\,\frac{\cos(2n-1)\,\theta}{r^{2n-1}}\right). \quad (3.8)$$

Therefore, when $r = r_0(x)$,

$$\frac{\partial\phi}{\partial r} = a_0(x)\left\{-\frac{2}{r_0} + 2K\ln\frac{r_0}{L}\cos\theta - 2K\,\theta\sin\theta\right\}$$
$$-2K\cos\theta\,\mathscr{L}(a_0)$$
$$-2\sum_{1}^{\infty}a_n(x)\left(\frac{\cos 2n\theta}{r_0^{2n+1}} + \frac{K}{2n}\,\frac{\cos(2n-1)\,\theta}{r_0^{2n}}\right)$$
$$= V(x)\cos\theta \quad \text{from the approximate boundary condition (2.4).} \quad (3.9)$$

We see immediately that $a_0 = O(Vr_0)$ and $a_n = O(Vr_0^{2n+1})$. Let us find the potential (and therefore the pressure) on the body correct to order

$$Vr_0\,Kr_0\ln(K/r_0)$$

when KL is small or moderate. Since the leading term in ϕ is of order $Vr_0\ln(L/r_0)$ we see that in that term we need $a_0(x)$ correct to order $Vr_0\,Kr_0$, whereas the coefficient functions $a_n(x)$ are needed only to order $Vr_0^{2n+1}Kr_0\ln(L/r_0)$. These functions are found by a Fourier method. When (3.9) is integrated from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, we obtain

$$a_0(x)\left\{-\frac{2}{r_0}\tfrac{1}{2}\pi - 2K\ln\frac{L}{r_0}\right\} - 2Ka_0(x) - 2K\mathscr{L}(a_0)$$
$$-2K\sum_{1}^{\infty}\frac{a_n(x)}{r_0^{2n}}\,\frac{(-1)^{n-1}}{2n(2n-1)} = V(x); \quad (3.10)$$

when (3.9) is multiplied by $\cos 2n\theta$ ($n = 1, 2, 3, \ldots$) and integrated from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, we obtain

$$a_0(x)\left\{-2K\frac{(-1)^{n-1}}{4n^2-1}\ln\frac{L}{r_0} + O(K)\right\} - 2\frac{a_n(x)}{r_0^{2n+1}}\tfrac{1}{4}\pi$$
$$= V(x)\frac{(-1)^{n-1}}{4n^2-1}, \quad (3.11)$$

on noting that

$$\cos\theta = \frac{4}{\pi}\left(\frac{1}{2}+\sum_{1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\cos 2n\theta\right). \tag{3.12}$$

In (3.11) it is sufficient to use the crude approximation $a_0(x) \doteqdot -\pi^{-1}Vr_0$. Then from (3.11),

$$a_n(x) = \frac{2}{\pi}\frac{(-1)^n}{4n^2-1}\frac{Vr_0^{2n+1}}{1+(2/\pi)Kr_0\ln(L/r_0)} \tag{3.13}$$

$$= \frac{2(-1)^n}{4n^2-1}W(x)\,r_0^{2n}, \quad \text{say,} \tag{3.14}$$

with an error of order $Vr_0^{2n+1}Kr_0$, where by definition

$$W(x) = \frac{1}{\pi}\frac{V(x)\,r_0(x)}{1+(2/\pi)Kr_0\ln(L/r_0)} \doteqdot \frac{1}{\pi}V(x)\,r_0(x)-\frac{2}{\pi^2}V(x)\,r_0(x)\,Kr_0\ln\frac{L}{r_0}. \tag{3.15}$$

In the terms on the second and third line of (3.10) it is sufficient to write

$$a_0(x) \doteqdot -\pi^{-1}Vr_0, \quad \text{and} \quad a_n(x) \doteqdot \frac{2}{\pi}\frac{(-1)^n}{4n^2-1}Vr_0^{2n+1}, \quad \text{respectively,}$$

thus

$$-\frac{a_0(x)\,\pi}{r_0}\left(1+\frac{2}{\pi}Kr_0\ln\frac{L}{r_0}\right) = V(x)-\frac{2}{\pi}VKr_0-\frac{2}{\pi}K\mathscr{L}(Vr_0) -\frac{4K}{\pi}Vr_0\sum_{1}^{\infty}\frac{1}{2n(2n-1)(4n^2-1)}, \tag{3.16}$$

whence

$$a_0(x) = -W(x)+\frac{2}{\pi}Kr_0\mathscr{L}(W)+\frac{2}{\pi}Kr_0\,W(x)\left\{1+2\sum_{1}^{\infty}\frac{1}{2n(2n-1)(4n^2-1)}\right\}. \tag{3.17}$$

On substituting (3.14) and (3.17) in the expression (3.8) for the potential we find that on the body, correct to order $Vr_0Kr_0\ln(L/r_0)$, the potential corresponding to the radial-velocity boundary condition is

$$\phi(x, r_0\cos\theta, r_0\sin\theta) = 2a_0(x)\,(1-Kr_0\cos\theta)\ln\frac{L}{r_0}-2\mathscr{L}(W) + 4W(x)\sum_{1}^{\infty}\frac{(-1)^n}{2n(4n^2-1)}\cos 2n\theta \tag{3.18}$$

$$= -2W(x)\,(1-Kr_0\cos\theta)\ln\frac{L}{r_0}-2\mathscr{L}(W)\left(1-\frac{2}{\pi}Kr_0\ln\frac{L}{r_0}\right) +\frac{4}{\pi}W(x)\,Kr_0\ln\frac{L}{r_0}\left\{1+2\sum_{1}^{\infty}\frac{1}{2n(2n-1)(4n^2-1)}\right\} +2W(x)\sum_{1}^{\infty}\frac{(-1)^n}{n(4n^2-1)}\cos 2n\theta, \tag{3.19}$$

where $W(x)$ is defined by (3.15), and $\mathscr{L}$ denotes the linear operator defined by (3.7). The terms involving $\mathscr{L}(W)$ contain integrals of W over the whole length of the body and involve values of V and r_0 in cross-sections other than x; thus $\mathscr{L}(W)$ expresses the interaction between sections. The order of magnitude of this interaction term is only slightly less (by a factor $\ln(L/r_0)$) than the leading term in the potential. This was expected since slender-body theory for the limit $KL \to 0$ leads to a similar result, see equation (3.6).

We must, however, remember that the exact boundary condition (see equation (2.3)) is

$$\partial\phi/\partial r = V(x)\cos\theta + r_0'(x)\,\partial\phi/\partial x, \tag{3.20}$$

and in the last term on the right we may substitute the leading term

$$\phi \doteqdot -2W(x)\ln(L/r_0).$$

We then find that the modification to the radial velocity is of order

$$V_0(r_0/L)^2\ln(L/r_0)$$

and therefore negligible, and (3.19) is thus also the solution of the problem with boundary condition (3.20) correct to our order of approximation.

4. The boundary-value problem for moderate and large values of KL

This will be treated more briefly since the analysis is similar to § 3 above. To avoid confusion the values of the coefficient functions are denoted by $a_0^*(x)$ and $a_n^*(x)$, and we use the expression (2.44)–(2.47) for the potential, except that we substitute (2.31) in (2.45). Thus

$$\phi(x,y,z) = a_0^*(x)\{-2\ln Kr + 2Kr\ln Kr\cos\theta - 2Kr\,\theta\sin\theta - 2Kr\cos\theta\} \tag{4.1}$$

$$+\,2(1-Kr\cos\theta)\,\mathscr{L}^*(a_0^*) \tag{4.2}$$

$$+\,2\sum_1^\infty \frac{a_n^*(x)}{2n}\left\{\frac{\cos 2n\theta}{r^{2n}} + \frac{K}{2n-1}\,\frac{\cos(2n-1)\theta}{r^{2n-1}}\right\}, \tag{4.3}$$

where by definition

$$\mathscr{L}^*(a_0^*) = (i\pi-\gamma)\,a_0^*(x) - \frac{1}{2}\int_{-\infty}^{x}\frac{da_0^*(\xi)}{d\xi}s_1(K|x-\xi|)\,d\xi$$

$$+\frac{1}{2}\int_{-\infty}^{x}\frac{da_0^*(\xi)}{d\xi}s_1(K|x-\xi|)\,d\xi \tag{4.4}$$

$$-\frac{1}{2}\int_{-\infty}^{x}\frac{da_0^*(\xi)}{d\xi}s_2(K|x-\xi|)\,d\xi + \frac{1}{2}\int_{-\infty}^{x}\frac{da_0^*(\xi)}{d\xi}s_2(K|x-\xi|)\,d\xi. \tag{4.5}$$

The functions $s_1(\eta)$ and $s_2(\eta)$ are defined by (2.29) and (2.30), and by (2.36), respectively; both these functions are small when η is large. Thus for moderate or large KL we have $|\mathscr{L}^*(a_0^*)| = O\,|a_0^*|$. Comparison with (3.8) shows that the calculation is formally identical with the calculation of § 3 provided that we write K^{-1} for L and $\mathscr{L}^*$ for $\mathscr{L}$. We find that the potential on the body for moderate or large KL has the value

$$\begin{aligned}\phi(x, r_0\cos\theta, r_0\sin\theta) = {} & -2W^*(x)\,(1-Kr_0\cos\theta)\ln\frac{1}{Kr_0} \\ & -2\mathscr{L}^*(W^*)\left(1-\frac{2}{\pi}Kr_0\ln\frac{1}{Kr_0}\right) \\ & +\frac{4}{\pi}Kr_0\,W^*(x)\ln\frac{1}{Kr_0}\left\{1+2\sum_1^\infty\frac{1}{2n(2n-1)(4n^2-1)}\right\} \\ & +2W^*(x)\sum_1^\infty\frac{(-1)^n}{n(4n^2-1)}\cos 2n\theta,\end{aligned} \tag{4.6}$$

where

$$W^*(x) = \frac{1}{\pi}\frac{V(x)\,r_0(x)}{1+(2/\pi)\,Kr_0\ln(1/Kr_0)} \doteqdot \frac{1}{\pi}\,V(x)\,r_0(x) - \frac{2}{\pi^2}\,V(x)\,r_0(x)\,Kr_0\ln\frac{1}{Kr_0}. \quad (4.7)$$

We also have

$$a_0^*(x) = -W^*(x) + \frac{2}{\pi}Kr_0\,\mathscr{L}^*(W^*) + \frac{2}{\pi}Kr_0\,W^*\left\{1+2\Sigma\frac{1}{2n(2n-1)(4n^2-1)}\right\}. \quad (4.8)$$

The interaction between sections is given by the terms involving $\mathscr{L}^*$, and it can be shown that for moderate KL the potentials (3.19) and (4.6) are in agreement. We note from (4.4) and (4.5) that $\mathscr{L}^*(W^*) \to (i\pi-\gamma)\,W^*$ as $KL\to\infty$. Thus in the limit as $KL\to\infty$ we find that the potential (and thus the pressure) depends only on the local values of V and r_0; this is physically reasonable and a similar conclusion presumably holds whenever KL is larger, whether Kr_0 is small or not. The potential is then expected to be nearly the same as the two-dimensional potential on an infinite cylinder having radius $r_0(x)$, where the potential satisfies the wave-boundary condition

$$K\phi + \partial\phi/\partial y = 0 \quad \text{on} \quad y = 0$$

and

$$\partial\phi/\partial r = V_0(x)\cos\theta \quad \text{on} \quad r = r_0(x).$$

This problem has been discussed by Ursell (1949*a*, 1953, 1957).

As before (see the end of § 3) the solution is unchanged to our order of approximation when the exact boundary condition (2.3) is substituted for the approximate boundary condition (2.4).

5. Forces, moments and wave damping

We note that the element of area of any body of revolution $r = r_0(x)$ is given by $r_0\{1+(r_0')^2\}^{\frac{1}{2}}\,dx\,d\theta$. We observe also that the dominant forces and moments are hydrostatic. The hydrostatic force on a strip dx due to immersion $y_0(x)$ is $2\rho g r_0 y_0\,dx = 2i\rho\sigma V(x)\,r_0(x)\,K^{-1}\,dx$, the total hydrostatic moment about the z-axis is $2i\rho\sigma(x+r_0r_0')\,Vr_0K^{-1}\,dx$. These expressions are exact.

The total vertical component of hydrodynamical force on a strip is easily seen to be

$$2i\rho\sigma r_0(x)\,dx\int_0^{\frac{1}{2}\pi}\phi(x, r_0\cos\theta, r_0\sin\theta)\cos\theta\,d\theta, \quad (5.1)$$

and the total moment about the z-axis on a strip of width dx is

$$2i\rho\sigma r_0(x)\,(x+r_0r_0')\,dx\int_0^{\frac{1}{2}\pi}\phi(x, r_0\cos\theta, r_0\sin\theta)\cos\theta\,d\theta. \quad (5.2)$$

Both force and moment involve the expression $\int_0^{\frac{1}{2}\pi}\phi\cos\theta\,d\theta$. For instance, when (3.19) is substituted, this integral is

$$-2W(x)\ln\frac{L}{r_0}(1-\tfrac{1}{4}\pi Kr_0) - 2\mathscr{L}(W)\left(1-\frac{2}{\pi}Kr_0\ln\frac{L}{r_0}\right)$$

$$+\frac{4}{\pi}W(x)\,Kr_0\ln\frac{L}{r_0}\left\{1+2\sum_1^\infty\frac{1}{2n(2n-1)(4n^2-1)}\right\} \quad (5.3)$$

$$-2W(x)\sum_1^\infty\frac{1}{n(4n^2-1)^2}. \quad (5.4)$$

Now the brace in (5.3) can be shown to have the value $\frac{3}{2}-2\ln 2+\frac{1}{8}\pi^2$, and the sum in (5.4) to have the value $\frac{3}{2}-2\ln 2$. Thus

$$\int_0^{\frac{1}{2}\pi}\phi\cos\theta\,d\theta = -2W(x)\ln(L/r_0)+(4/\pi)\,W(x)\,Kr_0\ln(L/r_0)\,(\tfrac{3}{2}-2\ln 2+\tfrac{1}{4}\pi^2)$$
$$-2\mathscr{L}(W)\{1-(2/\pi)\,Kr_0\ln(L/r_0)\}-2W(x)\,(\tfrac{3}{2}-2\ln 2). \quad (5.5)$$

The expression derived from (4.6) is similar, with K^{-1} replacing L, W^* replacing W, and $\mathscr{L}^*$ replacing $\mathscr{L}$. If we neglect all terms of order smaller than Kr_0 times the hydrostatic force, we have

$$\int_0^{\frac{1}{2}\pi}\phi\cos\theta\,d\theta \doteqdot -2W(x)\ln(L/r_0)-W(x)\,(3-4\ln 2)-2\mathscr{L}(W). \quad (5.6)$$

Turning now to the wave damping, we find first the potential at a great distance from the body. The only term in the potential that contributes to this is the first term

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} a_0(\xi)\,\phi_0(x-\xi,y,z)\,d\xi, \quad (5.7)$$

and for large values of x^2+z^2 we may replace ϕ_0 by its asymptotic value. It is easily shown from (2.9), that

$$\phi_0(x,y,z) \sim i\pi K\,e^{-Ky}H_0^{(1)}\{K(x^2+z^2)^{\frac{1}{2}}\},$$

see also Havelock (1955). Then if $x=l\cos\alpha$, $z=l\sin\alpha$, the integral (5.7) is asymptotically

$$i\pi K\,e^{-Ky}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} a_0(\xi)\,H_0^{(1)}\{K[(x-\xi)^2+z^2]^{\frac{1}{2}}\}\,d\xi$$
$$= i\pi K\,e^{-Ky}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} a_0(\xi)\,H_0^{(1)}\{K[l^2-2l\xi\cos\alpha+\xi^2]^{\frac{1}{2}}\}\,d\xi \quad (5.8)$$
$$= i\pi K\,e^{-Ky}\Big[H_0^{(1)}(Kl)\int_{-\frac{1}{2}L}^{\frac{1}{2}L} a_0(\xi)\,J_0(K\xi)\,d\xi$$
$$+2\sum_{n=1}^{\infty}H_n^{(1)}(Kl)\cos n\alpha\int_{-\frac{1}{2}L}^{\frac{1}{2}L} a_0(\xi)\,J_n(K\xi)\,d\xi\Big] \quad (5.9)$$

by a known addition theorem (Erdélyi 1953, p. 101, equation (29)) for the Hankel function. Equation (5.9) gives explicitly the wave amplitude contained in each individual Fourier component. The mean energy radiated per unit time by the nth component is obtained from the asymptotic expansion

$$H_n^{(1)}(Kl) \sim (2/\pi Kl)^{\frac{1}{2}}\,e^{iKl}e^{-\frac{1}{2}in\pi-\frac{1}{4}i\pi},$$

and from the result that the rate of energy transmission for a *real-valued* potential $\phi^*(x,y,z,t)$ is $-\rho(\partial\phi^*/\partial t)\,(\partial\phi^*/\partial n)$ per unit area, where $\partial\phi^*/\partial n$ is the velocity component normal to the area. We consider the energy transmission across a vertical circular cylinder of large radius. On taking the real and imaginary parts of (5.9) we find in this way that the mean rate of energy transmission for the nth Fourier component is

$$2\rho\sigma\pi^2K\left|\int_{-\frac{1}{2}L}^{\frac{1}{2}L} a_0(\xi)\,J_n(K\xi)\,d\xi\right|^2$$
$$= 2\rho\sigma\pi^2K\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} a_0(\xi)\,\overline{a_0(\xi')}\,J_n(K\xi)\,J_n(K\xi')\,d\xi\,d\xi' \quad (5.10)$$

when $n \geqslant 1$, and is half this expression when $n = 0$, where $\overline{a_0}$ is the complex conjugate of a_0. Also it is easy to show by direct integration that the various Fourier components radiate energy independently. The total mean rate of energy radiation is therefore

$$\begin{aligned} E &= \pi^2\rho\sigma K \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi' a_0(\xi)\, \overline{a_0(\xi')} \left[J_0(K\xi)\, J_0(K\xi') + 2\sum_1^{\infty} J_n(K\xi)\, J_n(K\xi') \right] \\ &= \pi^2\rho\sigma K \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi' a_0(\xi)\, \overline{a_0(\xi')}\, J_0(K\,|x-\xi|), \end{aligned} \tag{5.11}$$

by the addition theorem for Bessel functions; in this expression $a_0(\xi)$ is given by (3.17) or (4.8) according as KL is not large or not small. An equivalent expression in terms of Fourier transforms is

$$\begin{aligned} E &= \pi^2\rho\sigma K \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \overline{a_0(\xi')}\, d\xi' \frac{1}{2\pi} \int_{-K}^{K} A_0(k) \frac{2}{(K^2-k^2)^{\frac{1}{2}}} e^{-ikx}\, dk \\ &= \pi\rho\sigma K \int_{-K}^{K} |A_0(k)|^2 \frac{dk}{(K^2-k^2)^{\frac{1}{2}}} \\ &= \pi\rho\sigma K \int_0^{\pi} |A_0(K\cos\beta)|^2\, d\beta. \end{aligned} \tag{5.12}$$

6. Conclusion

We have now obtained the velocity potential of a slender pointed body of revolution making small oscillations in the free surface at zero forward speed. Experience with slender-body theory in an incompressible unbounded medium suggests that these results can be extended to more general cross-sections (cf. Ursell 1949*b*), and since the problem remains linear (at zero speed) it may also be possible to allow for the effect of blunt ends. Much more interesting would be a solution to the full problem of the oscillating slender body at forward speed (necessarily pointed if the disturbance is to be small), for here our present knowledge is very incomplete. Like slender-body theory in a compressible unbounded medium (Ward 1955) this gives rise to non-linear problems for which a scheme of successive approximations based on a slenderness parameter can be developed. The nth approximation satisfies linear equations involving the previous $(n-1)$ approximations, and as soon as these are involved non-linearly the calculation in practice terminates. It will be interesting to see whether these calculations can be carried far enough to give useful results for the full problem. Recent work (Tuck 1962) has already shown that a useful theory can be constructed on these lines for non-oscillating surface ships at forward speed.

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Appendix. Fourier analysis of wave-source and wave-free potentials

A. 1. *The wave source at the origin*

According to (2.9) the potential is

$$\phi_0(x,y,z) = \int_0^\infty \frac{k'}{k'-K} e^{-k'y} J_0\{k'(x^2+z^2)^{\frac{1}{2}}\}\,dk'$$

$$= \int_0^\infty \frac{k'}{k'-K} e^{-k'y} \left\{\frac{1}{\pi}\int_0^\pi e^{ik'x\cos\beta'}\cos(k'z\sin\beta')\,d\beta'\right\} dk'. \quad (A.1.1)$$

In the inner integral change the variable of integration to $k = -k'\cos\beta'$. Then this integral is

$$\frac{1}{\pi}\int_{-k'}^{k'} e^{-ikx}\cos\{z(k'^2-k^2)^{\frac{1}{2}}\}\,dk,$$

and on changing the order of integration by use of the relation

$$\int_0^\infty \left(\int_{-k'}^{k'} f(k,k')\,dk\right) dk' = \int_{-\infty}^\infty \left(\int_{|k|}^\infty f(k,k')\,dk'\right) dk$$

it is found that

$$\phi(x,y,z) = \int_{-\infty}^\infty e^{-ikx}\left(\frac{1}{\pi}\int_{|k|}^\infty \frac{k'}{k'-K} e^{-k'y}\frac{\cos\{z(k'^2-k^2)^{\frac{1}{2}}\}}{(k'^2-k^2)^{\frac{1}{2}}}\,dk'\right) dk. \quad (A.1.2)$$

It follows at once that the Fourier transform

$$\Phi_0(k,y,z) = 2\int_{|k|}^\infty \frac{k'}{k'-K} e^{-k'y}\frac{\cos\{z(k'^2-k^2)^{\frac{1}{2}}\}}{(k'^2-k^2)^{\frac{1}{2}}}\,dk' \quad (A.1.3)$$

$$= 2\int_0^\infty \frac{|k|\cosh\mu}{|k|\cosh\mu - K} e^{-|k|y\cosh\mu}\cos(|k|z\sinh\mu)\,d\mu. \quad (A.1.4)$$

The expansion (2.13) of this integral for small $|k|\,r$ will now be proved. There are two different expressions, according as $K \gtrless |k|$. If $K > |k|$, put $K = |k|\cosh\alpha$. It will first be shown that, when $z = 0$,

$$\Phi_0(k,r,0) = (\pi i - \alpha)\coth\alpha\left(2I_0(|k|\,r) + 4\sum_1^\infty(-1)^m I_m(|k|\,r)\cosh m\alpha\right)$$

$$+ 2K_0(|k|\,r) + 4\sum_1^\infty(-1)^{m-1}\left[\frac{\partial}{\partial\nu}I_\nu(|k|\,r)\right]_{\nu=m}\sinh m\alpha\coth\alpha. \quad (A.1.5)$$

For let β be an arbitrary positive number and consider the Laplace transform of Φ_0,

$$\int_0^\infty \Phi_0(k,r,0)\,e^{-|k|r\cosh\beta}\,dr = 2\int_0^\infty \frac{\cosh\mu}{\cosh\mu-\cosh\alpha}\left(\int_0^\infty e^{-|k|r(\cosh\beta+\cosh\mu)}\,dr\right) d\mu$$

$$= \frac{2}{|k|}\int_0^\infty \frac{\cosh\mu\,d\mu}{(\cosh\mu-\cosh\alpha)(\cosh\mu+\cosh\beta)}$$

$$= \frac{2}{|k|}\int_0^\infty \left(\frac{\cosh\alpha}{\cosh\mu-\cosh\alpha} + \frac{\cosh\beta}{\cosh\mu+\cosh\beta}\right)\frac{d\mu}{\cosh\beta+\cosh\alpha},$$

an elementary integral,

$$= \frac{2}{|k|}\frac{1}{\cosh\beta+\cosh\alpha}\{(\pi i-\alpha)\coth\alpha+\beta\coth\beta\}$$

$$= \frac{2}{|k|}\frac{(\pi i-\alpha)\coth\alpha}{\sinh\beta}\left(1+2\sum_1^\infty(-1)^m\cosh m\alpha\, e^{-m\beta}\right)$$

$$+\frac{\beta}{\sinh\beta}\left(1+2\sum_1^\infty(-1)^m\sinh m\alpha\coth\alpha\, e^{-m\beta}\right). \tag{A. 1.6}$$

And it is known (Watson 1944, p. 386, equation (8)) that

$$\int_0^\infty e^{-|k|r\cosh\beta}I_\nu(|k|\,r)\,dr = \frac{e^{-\nu\beta}}{|k|\sinh\beta}, \tag{A. 1.7}$$

whence

$$\int_0^\infty e^{-|k|r\cosh\beta}\frac{\partial I_\nu(|k|\,r)}{\partial\nu}\,dr = -\frac{\beta e^{-\nu\beta}}{|k|\sinh\beta}, \tag{A. 1.8}$$

and in particular

$$\int_0^\infty e^{-|k|r\cosh\beta}K_0(|k|\,r)\,dr = \frac{\beta}{|k|\sinh\beta}. \tag{A. 1.9}$$

On applying the inverse Laplace transformation to (A. 1.6) and using (A. 1.7)–(A. 1.9) the result (A. 1.5) is obtained. To obtain the expansion (2.11) of $\Phi_0(k,y,z)$ when $z \neq 0$ we observe that Φ_0 is a solution of the elliptic partial differential equation

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k^2\right)\Phi_0(k,y,z) = 0 \tag{A. 1.10}$$

and is thus completely determined by its values on $z = 0$, since it is an even function of z. But the series on the right-hand side of (2.11) is also clearly an even solution of (A. 1.10) since all its terms are even solutions, and by (A. 1.5) it coincides with Φ_0 when $z = 0$. This establishes equation (2.11) when $K > |k|$. When $K < |k|$ the argument is similar but simpler since (A. 1.4) can then be integrated along the real μ-axis.

A. 2. *Wave-free potentials at the origin*

We have

$$\frac{P_m(y/R)}{R^{m+1}} = \frac{1}{m!}\int_0^\infty (k')^m e^{-k'y}J_0\{k'(x^2+z^2)^{\frac{1}{2}}\}\,dk', \tag{A. 2.1}$$

since both sides are functions harmonic in $y > 0$ and take the value y^{-m-1} on the y-axis. Thus

$$\phi_n(x,y,z) = \frac{1}{(2n)!}\int_0^\infty (K+k')(k')^{2n-1}e^{-k'y}J_0\{k'(x^2+z^2)^{\frac{1}{2}}\}\,dk' \tag{A. 2.2}$$

whence it follows, as in § A. 1, that

$$\Phi_n(k,y,z) = \frac{2}{(2n)!}\int_{|k|}^\infty (K+k')(k')^{2n-1}e^{-k'y}\frac{\cos\{z(k'^2-k^2)^{\frac{1}{2}}\}}{(k'^2-k^2)^{\frac{1}{2}}}\,dk'. \tag{A. 2.3}$$

This is equivalent to equation (2.40) above.

REFERENCES

Erdélyi, A. (ed.) 1953 *Higher Transcendental Functions*, vol. 2, *Bateman Manuscript Project*. New York: McGraw-Hill.

Grim, O. 1957 Durch Wellen an einem Schiffskörper erregte Kräfte. *Proc. Symp. on the Behaviour of Ships in a Seaway*, pp. 232–65. Wageningen: H. Veenman and Zonen.

Grim, O. 1960 A method for a more precise computation of heaving and pitching motions both in smooth water and in waves. *Third Sym. on Naval Hydrodynamics, Scheveningen*, pp. 271–326. Wageningen: H. Veenman and Zonen.

Havelock, Sir Thomas 1955 Waves due to a floating sphere making periodic heaving oscillations. *Proc. Roy. Soc.* A, **231**, 1–7.

Havelock, Sir Thomas 1956 The damping of heave and pitch: a comparison of two-dimensional and three-dimensional calculations. *Trans. Inst. Nav. Archit., Lond.*, **98**, 464–8.

Lighthill, M. J. 1958 *Fourier Analysis and Generalised Functions*. Cambridge University Press.

Michell, J. H. 1898 The wave resistance of a ship. *Phil. Mag.* (5) **45**, 106–23.

Peters, A. S. & Stoker, J. J. 1957 The motion of a ship, as a floating rigid body, in a seaway. *Comm. Pure Appl. Math.* **10**, 399–490.

Tasai, F. 1959 On the damping force and added mass of ships heaving and pitching. *J. Zosen Kiokai* (*Soc. Naval Architects of Japan*), **105**, 47–56. Translation published by *Inst. Engng Res., Univ. of California, Berkeley*, Series 82, Issue 15, 1960.

Thorne, R. C. 1953 Multipole expansions in the theory of surface waves. *Proc. Camb. Phil. Soc.* **49**, 707–16.

Thwaites, B. (ed.) 1960 *Incompressible Aerodynamics*. Oxford University Press.

Tuck, E. O. 1962 Unpublished.

Ursell, F. 1949*a* On the heaving motion of a circular cylinder on the surface of a fluid. *Quart. J. Mech. Appl. Math.* **2**, 218–31.

Ursell, F. 1949*b* On the rolling motion of cylinders in the surface of a fluid. *Quart. J. Mech. Appl. Math.* **2**, 335–53.

Ursell, F. 1953 Short surface waves due to an oscillating immersed body. *Proc. Roy. Soc.* A, **220**, 90–103.

Ursell, F. 1957 On the virtual mass and damping of floating bodies at zero speed ahead. *Proc. Symp. on the Behaviour of Ships in a Seaway*, pp. 374–87. Wageningen: H. Veenman and Zonen.

Vossers, G. 1962. *Some applications of the slender body theory in ship hydrodynamics*. Publication no. 214, Netherlands Ship Model Basin, Wageningen.

Ward, G. N. 1955 *Linearized Theory of Steady High-Speed Flow*, Ch. 9. Cambridge University Press.

Watson, G. N. 1944 *Bessel Functions*, 2nd ed. Cambridge University Press.

Wehausen, J. 1957 Wave resistance of thin ships. *Symp. on Naval Hydrodynamics, Washington, D.C.*, 1956, pp. 109–33. Washington: National Academy of Sciences—National Research Council.